The 14^{th} Romanian Master of Mathematics Competition

Day 1: Wednesday, March 1^{st}, 2023, Bucharest

Language: English

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying

$$x^3 + y^3 = p(xy + p).$$

Problem 2. Fix an integer $n \geqslant 3$. Let $\mathcal{S}$ be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in $\mathcal{S}$, the triangle ABC is *nice for* AB if $\text{Area}(ABC) \leqslant \text{Area}(ABX)$ for all X in $\mathcal{S}$ different from A and B. (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in $\mathcal{S}$ and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n-1)$ beautiful triangles.

Problem 3. Let $n \geqslant 2$ be an integer, and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \ldots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f.

(Note, for example, that the degree of the polynomial

$$g(x, y) = 4x^3y^4 + yx + x - 2$$

is 7 because $7 = 3 + 4$.)

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

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Day 2: Thursday, March 2^{nd}, 2023, Bucharest

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Problem 4. Given an acute triangle ABC, let H and O be its orthocentre and circumcentre, respectively. Let K be the midpoint of the line segment AH. Also let ℓ be a line through O, and let P and Q be the orthogonal projections of B and C onto ℓ, respectively.

Prove that $KP + KQ \geqslant BC$.

Problem 5. Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be non-constant polynomials with real coefficients such that $P(Q(x)) = R(S(x))$. Suppose that the degree of $P(x)$ is divisible by the degree of $R(x)$.

Prove that there is a polynomial $T(x)$ with real coefficients such that

$$P(x) = R(T(x)).$$

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

(A *spanning tree* of Γ is a graph which has the same vertices as Γ, with edges which are also edges of Γ, for which there is exactly one path between each pair of different vertices.)

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.